Class	Register Number	Name

南洋女子中学校

NANYANG GIRLS' HIGH SCHOOL

End-of-Year Examination 2015
Secondary Three

INTEGRATED MATHEMATICS 2 **2 hours**

Monday **5 Oct 2015** **0845 – 1045**

READ THESE INSTRUCTIONS FIRST

INSTRUCTIONS TO CANDIDATES

1. Write your name, register number and class in the spaces at the top of this page.
2. Answer questions 1 - 11 before attempting question 12 (Bonus Question).
3. Write your answers and working on the separate writing paper provided.
4. Omission of essential working will result in loss of marks.
5. Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION FOR CANDIDATES

1. The number of marks is given in brackets [ ] at the end of each question or part question.
2. The total number of marks for this paper is 80.
3. You are reminded of the need for clear presentation in your answers.

2

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} ab \sin C$$

1 **(a)** Find the range of values of x for which $(6-x)^2 > x$. [3]

(b) The roots of the quadratic equation $x^2 - 3x + 5 = 0$ are α and β. Find a quadratic equation with integer coefficients and with roots $\alpha^2 - \frac{1}{\beta}$ and $\beta^2 - \frac{1}{\alpha}$. [4]

2 Solve each of the following equations.

(a) $e^x - 7 = 2e^{-x}$ [4]

(b) $\log_9 y - 2 = \log_3 y$ [3]

3 **(a)** The function $y = f(x)$ undergoes two transformations:

I: Translation in the positive x-direction by 5 units,

II: Scaling in the y-direction by a factor of 2.

The final function is $y = 2x^2 - 16x + 32$. Find the function $f(x)$. [3]

(b) **(i)** Sketch the graph of $y = \ln(x-1)$, labelling clearly the intercept(s) and asymptote. [2]

(ii) By adding a suitable straight line to the graph in part **(b)(i)**, find the number of solutions to the equation $x = \frac{1}{e^{3-x}} + 1$. [2]

4 Solve the simultaneous equations

$27^x \div 3^y = 9,$

$2^{2x} \times 4^{1-y} = 64.$ [4]

5 The value, V dollars, of a vehicle depreciates over time. Given that $V = 84000e^{kt}$, where t is the time in years since it was bought and k is a constant, calculate

(i) the initial value of the vehicle, [1]

(ii) the value of k if, after 3 years, the value of the vehicle has halved, [2]

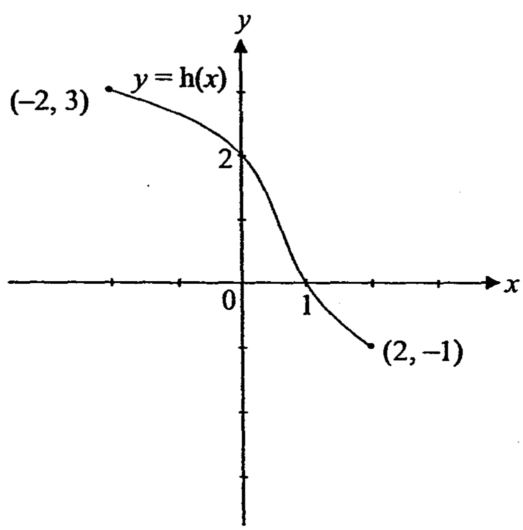
(iii) the value of t when the value of the vehicle is one-fifth its original value? [2]

6 **(a)** A curve has the equation $y^2 + (x + p)^2 = 8$, where p is a constant. Find the range of values of p for which the line $y = x + 4$ meets the curve. [4]

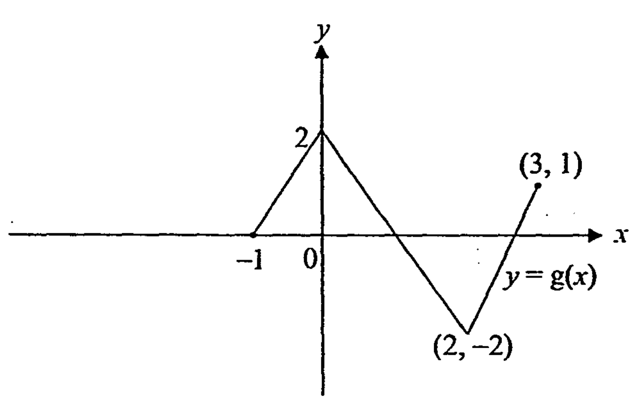
(b) Given that $y = x^2 - 4x + c$, find the value of the constant c for which the minimum value of y is 3. [3]

7 **Answer the whole of this question on the INSERT provided.**

(a) The graph of $y = \mathrm{h}(x)$ is given in the diagram. On the same axes shown on the **INSERT**, sketch the graph of $\mathrm{h}^{-1}(x)$. [3]



(b) The graph of $y = \mathrm{g}(x)$ is given in the diagram. On the same axes shown on the **INSERT**, sketch the graph of $y = \mathrm{g}(2x) + 1$. [2]



8 Given the function $y = -\sin\left(\frac{x}{2}\right) - 1$ for $x \geq 0$ radian.

(i) State the maximum and minimum value of y. [2]

(ii) State the period of y. [1]

(iii) State the amplitude of y. [1]

(iv) Find the smallest value of x such that $y = 0$. [2]

(v) Sketch the graph of $y = \left|-\sin\left(\frac{x}{2}\right) - 1\right|$ for $0 \leq x \leq 2\pi$. [2]

9 **(a)** Express the following in terms of sin θ, cos θ or tan θ, where θ is an acute angle.

(i) $\tan(2\pi + \theta)$, [1]

(ii) $\sin(2\pi - \theta)$, [1]

(iii) $\cos\left(\theta - \frac{\pi}{2}\right)$. [1]

(b) Solve the equation

$$\tan x + 2\sec^2 x - 5 = 0, \text{ for } 0 \leq x \leq 2\pi.$$ [5]

10 **(a)** Prove the identity $\operatorname{cosec} A - \cot A \equiv \frac{\sin A}{1 + \cos A}$. [3]

(b) Given that A and B are in different quadrants, $\tan A = -\frac{3}{4}$, $\cos B = -\frac{5}{13}$, $0° \leq A \leq 270°$ and $0° \leq B \leq 270°$. Without using a calculator, find the value of

(i) $\cos A$, [2]

(ii) $\tan B$, [2]

(iii) $\operatorname{cosec} A \sec B$, [2]

11 The functions f and g are defined by

$$f : x \mapsto \frac{3}{2x+1} \text{ for all values of } x \text{ except } x = -\frac{1}{2},$$

$$g : x \mapsto x^2 - 3x + 1.$$

(i) Find the values of x which map onto themselves under the function f. [3]

(ii) Find, in similar form, f^2. State the domain clearly. [3]

(iii) Express g(x) in the form of $h(x+k)^2 + n$ where h, k and n are constants. Hence, deduce the range of the function g. [3]

(iv) If the domain of g is $x \le c$ where c is a constant, state the maximum value of c for which the function g^{-1} exists. Hence, find the function g^{-1} in similar form. [4]

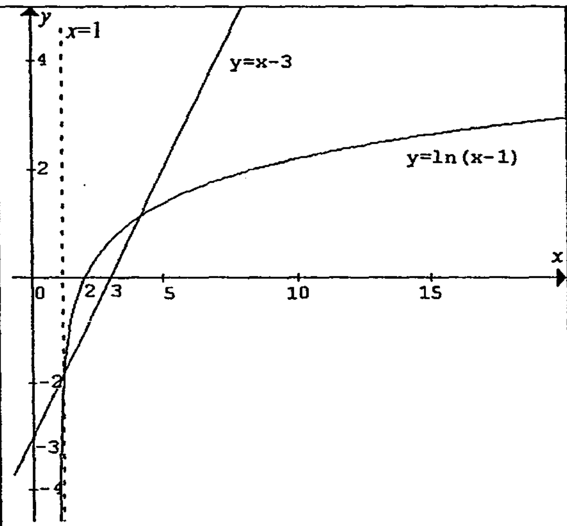
Bonus Question

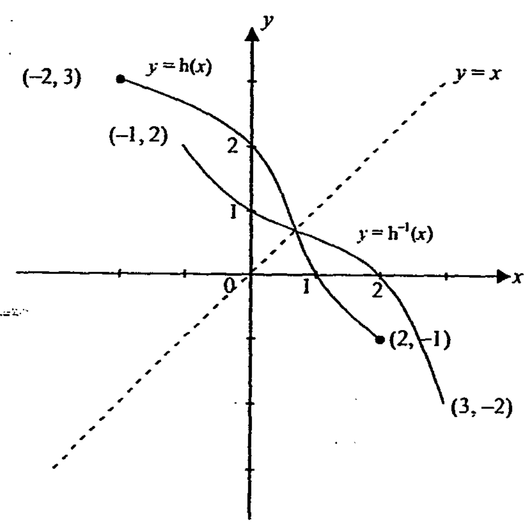
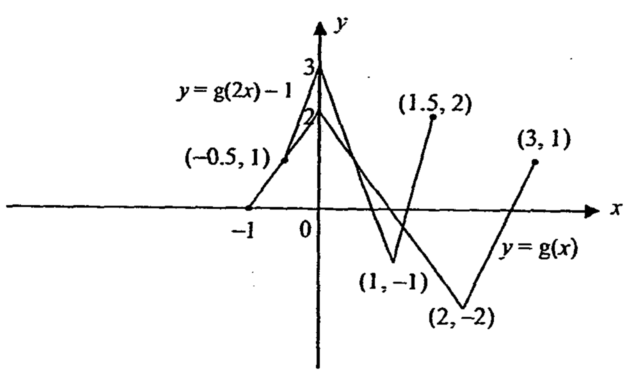
12 If $a > b > 1$ and $\frac{1}{\log_a b} + \frac{1}{\log_b a} = \sqrt{293}$, find the value of $\frac{1}{\log_{ab} b} - \frac{1}{\log_{ab} a}$. [3]

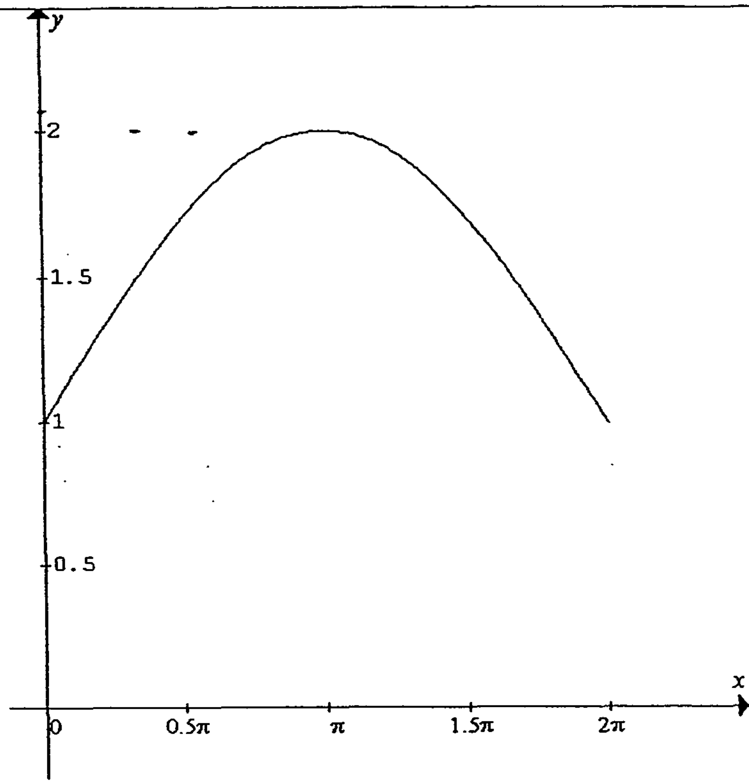
END OF PAPER

2015 Sec 3 IM2 EOY Mark Scheme

1a	$(6-x)^2 > x$ $36-12x+x^2 > x$ $x^2-13x+36>0$ $(x-9)(x-4)>0$ $\therefore x<4$ or $x>9$
1b	$\alpha+\beta=3,\ \alpha\beta=5$ $\alpha^2-\frac{1}{\beta}+\beta^2-\frac{1}{\alpha}=\alpha^2+\beta^2-\frac{1}{\alpha}-\frac{1}{\beta}$ $=(\alpha+\beta)^2-2\alpha\beta-\left(\frac{\alpha+\beta}{\alpha\beta}\right)$ $=9-10-\left(\frac{3}{5}\right)$ $=-\frac{8}{5}$ $\left(\alpha^2-\frac{1}{\beta}\right)\left(\beta^2-\frac{1}{\alpha}\right)=(\alpha\beta)^2-\alpha-\beta+\left(\frac{1}{\alpha\beta}\right)$ $=(\alpha\beta)^2-(\alpha+\beta)+\left(\frac{1}{\alpha\beta}\right)$ $=25-3+\left(\frac{1}{5}\right)$ $=\frac{111}{5}$ New equation: $x^2+\frac{8}{5}x+\frac{111}{5}=0$ $5x^2+8x+111=0$
2a	$e^{2x}-7(e^x)-2=0$ $(e^x)^2-7(e^x)-2=0$ $e^x=\frac{7\pm\sqrt{49-4(1)(-2)}}{2}$ $e^x=\frac{7+\sqrt{57}}{2}$ or $\frac{7-\sqrt{57}}{2}$ (NA) $\therefore x=\ln\frac{7+\sqrt{57}}{2}$ or 1.98
2b	$\frac{\log_3 y}{\log_3 9}-2=\log_3 y$ $\frac{1}{2}\log_3 y-2=\log_3 y$ $\frac{1}{2}\log_3 y=-2$ $\log_3 y=-4$ $\therefore y=\frac{1}{81}$

3a	$2f(x-5) = 2x^2 - 16x + 32$ $f(x-5) = x^2 - 8x + 16$ $= (x-4)^2$ $= (x+1-5)^2$ $\therefore f(x) = (x+1)^2$ *OR* Reverse 2nd transformation: $y = (2x^2 - 16x + 32) \div 2$ $= x^2 - 8x + 16$ Reverse 1st transformation: $y = (x+5)^2 - 8(x+5) + 16$ $= x^2 + 2x + 1$ $= (x+1)^2$
3bi	
3bii	$x = \frac{1}{e^{3-x}} + 1$ $x - 1 = e^{x-3}$ $\ln(x-1) = x - 3$ Draw the line $y = x - 3$ $\therefore$ number of solutions is 2
4	$27^x \div 3^y = 9 \rightarrow 3^{3x} \div 3^y = 3^2 \rightarrow 3x - y = 2$ ----(1) $2^{2x} \times 4^{1-y} = 64 \rightarrow 2^{2x} \times 2^{2-2y} = 2^6 \rightarrow x - y = 2$ ----(2) (or $2x - 2y = 4$) (1) – (2): $x = 0; y = -2$
5i	84000
5ii	$84000e^{3k} = 42000$ $e^{3k} = ½$ $3k = \ln ½$ $\therefore k = \frac{1}{3}\ln\frac{1}{2}$ (or –0.231)
5iii	$84000\ e^{kt} = 16800$

	$e^{kt} = 0.2$ $kt = \ln 0.2$ $\therefore t = 6.97$ (3sf)
6a	$(x+4)^2 + (x+p)^2 = 8$ $2x^2 + (8+2p)x + (8+p^2) = 0$ Eqn has real roots → Discriminant ≥ 0 $(8+2p)^2 - 4(2)(8+p^2) \geq 0$ $-4p^2 + 32p \geq 0$ $p^2 - 8p \leq 0$ OR $-p^2 + 8p \geq 0$ $p(p-8) \leq 0$ $p(8-p) \geq 0$ $\therefore 0 \leq p \leq 8$
b	$x^2 - 4x + c$ $= x^2 - 4x + 4 + c - 4$ $= (x-2)^2 + (c-4)$ $c - 4 = 3$ $\therefore c = 7$
7a	
7b	
8i	Max = 0, Min = − 2
8ii	4π

8iii	1.
8iv	$\sin\left(\frac{x}{2}\right)=-1$ $\frac{x}{2}=\frac{3\pi}{2}$ $\therefore x=3\pi$
8v	
9ai **ii** **iii**	$\tan\theta$ $-\sin\theta$ $\sin\theta$
9b	$\tan x+2(1+\tan^2 x)-5=0$ $2\tan^2 x+\tan x-3=0$ $(\tan x-1)(2\tan x+3)=0$ $\tan x=1$ or $\tan x=-\frac{3}{2}$ Ref. $\angle=\frac{\pi}{4}$ or 0.98279 $\therefore x=\frac{\pi}{4},\frac{5\pi}{4}$, 2.15 or 5.30
10a	$\text{LHS}=\frac{1}{\sin A}-\frac{\cos A}{\sin A}$ $=\frac{1-\cos A}{\sin A}$

	$= \frac{1-\cos A}{\sin A} \times \frac{1+\cos A}{1+\cos A}$ $= \frac{1-\cos^2 A}{\sin A(1+\cos A)}$ $= \frac{\sin^2 A}{\sin A(1+\cos A)}$ $= \frac{\sin A}{1+\cos A}$ = RHS (shown)
10bi	A in 2nd quadrant, hyp = 5 $\cos A = -\frac{4}{5}$
10bii	B in 3rd quadrant, opp = 12 $\tan B = \frac{12}{5}$
10biii	$\operatorname{cosec} A \sec B = \frac{1}{\sin A} \times \frac{1}{\cos B}$ $= \left(\frac{5}{3}\right)\left(-\frac{13}{5}\right)$ $= -\frac{13}{3}$ or $-4\frac{1}{3}$
11i	$\frac{3}{2x+1} = x$ $2x^2 + x = 3$ $2x^2 + x - 3 = 0$ $(2x+3)(x-1) = 0$ $\therefore x = -\frac{3}{2}$ or 1
11ii	$f^2(x) = f(\frac{3}{2x+1})$ $= \frac{3}{2\left(\frac{3}{2x+1}\right)+1}$ $= \frac{3}{\frac{6+2x+1}{2x+1}}$ $= \frac{3(2x+1)}{2x+7}$ or $\frac{6x+3}{2x+7}$ $f^2 : x \mapsto \frac{3(2x+1)}{2x+7}$ where $x \neq -\frac{7}{2}, x \neq -\frac{1}{2}$
11iii	$g(x) = x^2 - 3x + 1$

	$=\left(x-\frac{3}{2}\right)^2+1-\frac{9}{4}$ $=\left(x-\frac{3}{2}\right)^2-\frac{5}{4}$ Turning point is $\left(1\frac{1}{2},-1\frac{1}{4}\right)$ The range of g is $g(x)\ge -1\frac{1}{4}$
11iv	Max. $k=1\frac{1}{2}$ Let $y=g(x)$ $y=\left(x-\frac{3}{2}\right)^2-\frac{5}{4}$ $\left(x-\frac{3}{2}\right)^2=y+\frac{5}{4}$ $x-\frac{3}{2}=\pm\sqrt{y+\frac{5}{4}}$ $x=\pm\sqrt{y+\frac{5}{4}}+\frac{3}{2}$ $g^{-1}(x)=\pm\sqrt{x+\frac{5}{4}}+\frac{3}{2}$ Let $y=g(x)$ $y=x^2-3x+1$ $x^2-3x+1-y=0$ $x=\frac{3\pm\sqrt{9-4(1-y)}}{2}$ $x=\frac{3\pm\sqrt{5+4y}}{2}$ $g^{-1}(x)=\frac{3\pm\sqrt{5+4x}}{2}$ Let $g^{-1}(x)=y$ $x=g(y)$ $x=\left(y-\frac{3}{2}\right)^2-\frac{5}{4}$ $\left(y-\frac{3}{2}\right)^2=x+\frac{5}{4}$ $y-\frac{3}{2}=\pm\sqrt{x+\frac{5}{4}}$ $y=\pm\sqrt{x+\frac{5}{4}}+\frac{3}{2}$ Since $g^{-1}(x)\le 1\frac{1}{2}$, $g^{-1}: x\mapsto -\sqrt{x+\frac{5}{4}}+\frac{3}{2}\ ,x\ge -\frac{5}{4}$ or $\frac{3-\sqrt{5+4x}}{2}$
	Bonus qn $\frac{1}{\log_{ab}b}-\frac{1}{\log_{ab}a}=\log_b ab-\log_a ab$ $=(\log_b a+1)-(\log_a b+1)$ $=\log_b a-\log_a b$ $=\frac{1}{\log_a b}-\frac{1}{\log_b a}$ $=\sqrt{\left(\frac{1}{\log_a b}-\frac{1}{\log_b a}\right)^2}$ $=\sqrt{\left(\frac{1}{\log_a b}+\frac{1}{\log_b a}\right)^2-4\left(\frac{1}{\log_a b}\right)\left(\frac{1}{\log_b a}\right)}$ $=\sqrt{293-4}$ $=17$